Math 1151, Fall 2009

Final Exam

Recitation T.A____________ Section#:__________ I.D.#__________________

Name(Print):__________________________________________________________

Signature:___________________________________________________________

READ AND FOLLOW THESE INSTRUCTIONS

This booklet contains 13 pages. Check to see if any are missing. Print on the top all
the requested information, and sign your name. Put your name on the top of every page, in
case the pages become separated. Do your work in the blank spaces and back of pages of
this booklet.

This is a 3-hour examination consisting of 20 questions divided into two parts, worth a
total of 200 points. Part I consists of 14 machine-graded multiple-choice questions worth 8
points each (total of 112 points). Part II consists of 6 hand-graded questions worth 14 or 15
points points each (total of 88 points).

You may use a scientific calculator. You may not use a graphing calculator. You may
not use books or notes. SHOW ALL YOUR WORK! In part II, unsupported answers will
receive little credit.

Machine graded part: You will receive one answer sheet to record your answers for the
machine-graded portion of the exam. You MUST use a soft pencil (No. 1 or No. 2) to
answer this part. Do not fold or tear the answer sheet, and carefully enter all the requested
information according to the instructions you received. Do not make any stray marks on
the answer sheet. When you have decided on a correct answer to a given question, circle the answer in this booklet and blacken completely the corresponding circle in the answer
sheet. If you erase something, do so thoroughly. Each question has one and only one correct
answer. If you give two different answers, the question will be marked wrong. There is no
penalty for guessing, but if you don’t answer a question, skip the corresponding line in the
answer sheet. Go on to the next question.

Either the student or the School of Mathematics may for any reason request a regrade
of the machine graded part. All regrades will be based on responses in the test booklet,
and not on the machine graded response sheet. Any problem for which the answer is not
indicated in the test booklet, or which has no relevant accompanying calculations will be
marked wrong on the regrade. Therefore, work and answers must be clearly shown on the
test booklet.

After you finish both parts of the exam: Place the answer sheet between two pages
of this booklet (make a sandwich), with the side marked “General Purpose Answer Sheet”
facings DOWN. Have your ID card in your hand when handing in your exam.

Hand Graded Part:  15.— 16.— 17.— 18.— 19.— 20.—

Multiple Choice Part:

Total ———
Trigonometry
\[\sin^2 \theta + \cos^2 \theta = 1\]
\[\tan^2 \theta + 1 = \sec^2 \theta\]
\[1 + \cot^2 \theta = \csc^2 \theta\]

\[
\cos(\pi/2 - \theta) = \cos \theta \\
\cos(\pi/2 - \theta) = \sin \theta \\
\tan(\pi/2 - \theta) = \cot \theta
\]

\[
\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta
\]

\[
\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\
\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}
\]

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta
\]

\[
\sin 2\theta = 2 \sin \theta \cos \theta \\
\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}
\]

\[
\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \\
\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \\
\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}
\]

\[
\sin \frac{A}{a} = \sin \frac{B}{b} = \sin \frac{C}{c} \\
a^2 = b^2 + c^2 - 2bc \cos A
\]

Area of triangle: \[A = \frac{ab \sin C}{2} = \sqrt{s(s - a)(s - b)(s - c)}\] where \(s = (a + b + c)/2\)

Conic Sections
A parabola with vertex \((0, 0)\), focus \((a, 0)\) and directrix \(x = -a\) has equation \(y^2 = 4ax\).

An ellipse with vertices at \((-a, 0)\) and \((a, 0)\), minor axis of length \(2b\) has equation \[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\] and foci at \((-c, 0)\), \((c, 0)\) where \(c^2 = a^2 - b^2\).

A hyperbola with vertices at \((-a, 0)\), \((a, 0)\) and foci at \((-c, 0)\), \((c, 0)\) has equation \[\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\] where \(c^2 = a^2 + b^2\). Asymptotes: \(y = \frac{b}{a} x\) and \(y = -\frac{b}{a} x\).

Sequences and Summation
\[
\sum_{k=1}^{n} c = cn \\
\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \\
\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \\
\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2
\]

The arithmetic sequence with general term \(a_n = a_1 + (n - 1)d\) has sum
\[
S_n = a_1 + a_2 + \cdots + a_n = \frac{n}{2} [a_1 + a_n] = \frac{n}{2} [2a_1 + (n - 1)d].
\]

The geometric sequence with general term \(a_n = a_1 r^{n-1}\) has sum
\[
S_n = a_1 + a_1 r + \cdots + a_1 r^{n-1} = a_1 \cdot \frac{1 - r^n}{1 - r}, \quad r \neq 1.
\]

If \(|r| < 1\) then
\[
\sum_{k=1}^{\infty} a_1 r^{k-1} = \frac{a_1}{1 - r}.
\]
Part I, 14 questions worth 8 points each.

1. The length of arc of a circle of radius 3 subtended by a central angle of 40° is
   a. $120\pi$
   b. $2\pi/3$
   c. $\pi/3$
   d. $54\pi$
   e. $2\pi$

2. The expression $f(t) = \sin t - \cos t$ is equal to
   a. $\sin(t + \frac{\pi}{4})$
   b. $\sqrt{2} \sin(t + \frac{\pi}{4})$
   c. $\sqrt{2} \cos(t - \frac{3\pi}{4})$
   d. $\sqrt{2} \cos(t + \frac{3\pi}{4})$
   e. $2 \cos(t + \frac{3\pi}{4})$

3. If $\theta$ is an angle with $\pi \leq \theta \leq 2\pi$ for which $\sec \theta = \frac{13}{12}$ then $\tan \theta$ is
   a. $-\frac{12}{5}$
   b. $-\frac{5}{12}$
   c. $\frac{5}{12}$
   d. $\frac{7}{12}$
   e. $\frac{12}{5}$
4. The equation \(2\sin^2 \phi - \cos \phi - 1 = 0\) has
   a. No solutions in the interval \([0, 2\pi]\)
   b. One solution in the interval \([0, 2\pi]\)
   c. Two solutions in the interval \([0, 2\pi]\)
   d. Three solutions in the interval \([0, 2\pi]\)
   e. Four solutions in the interval \([0, 2\pi]\)

5. Express the equation \(r^2 = \cos 2\theta\) in rectangular form.
   a. \(x^2 + y^2 = 1 - 2y^2\)
   b. \(x^2 + y^2 = 2x^2 - 1\)
   c. \(x^2 + y^2 = x - y\)
   d. \((x^2 + y^2)^2 = 2xy\)
   e. \((x^2 + y^2)^2 = x^2 - y^2\)

6. The value of the complex number \(\frac{-5+4i}{4+5i}\) is
   a. \(i\)
   b. \(-\frac{41}{9}i\)
   c. \(-\frac{40}{41} + i\)
   d. \(-\frac{40}{41} - \frac{9}{41}i\)
   e. \(-\frac{40}{41} - \frac{41}{9}i\)
7. The complex number \((2(\cos 45^\circ + i \sin 45^\circ))^5\) is equal to
   a. \(-32\)
   b. \(-32i\)
   c. \(-16\sqrt{2} + 16\sqrt{2}i\)
   d. \(-16\sqrt{2} - 16\sqrt{2}i\)
   e. \(-32 - 32i\)

8. If \(a_1 = 14\) and \(a_n = \frac{1}{2}a_{n-1} + 3\) then \(a_1 + a_2 + a_3\) equals
   a. 6
   b. 12
   c. 22.5
   d. 32
   e. 42

9. If \(2x^3 + ax^2 + bx + c\) is a polynomial whose roots include 1 and \(-1 - i\) and in which \(a, b\) and \(c\) are real, then the coefficient \(b\) is
   a. \(-4\)
   b. \(-2\)
   c. 0
   d. 2
   e. 4
10. The polynomial $x^{12} - 2x^7 + 5x^4 + 3$ divided by $(x + 1)$ has remainder
   a. 1
   b. 4
   c. 7
   d. 11
   e. $x + 3$

11. Find the sum of the arithmetic sequence $52 + 56 + 60 + 64 + \cdots + 448$
   a. 24000
   b. 24698
   c. 24750
   d. 24846
   e. 25000

12. Find an equation for the parabola with focus at $(2, -1)$ and directrix the line $y = 3$.
   a. $8(y - 1) = (x - 2)^2$
   b. $-8(y - 1) = (x - 2)^2$
   c. $(y + 1)^2 = -8(x - 2)$
   d. $y + 1 = 2(x - 2)^2$
   e. $8(y + 1) = x^2 + 2$
13. The angle of depression from the top of a lighthouse 120 feet above the sea to a rowboat on the sea is 60°. The distance of the boat to the base of the lighthouse is
   a. 40 feet
   b. $30\sqrt{3}$ feet
   c. $40\sqrt{3}$ feet
   d. 60 feet
   e. $60\sqrt{3}$ feet

14. Where is the center of the ellipse $3x^2 + y^2 + 12x - 2y + 4 = 0$?
   a. $(-2, 1)$
   b. $(-6, 1)$
   c. $(-\frac{2}{3}, 2)$
   d. $(2, -1)$
   e. $(1, -1)$
Part II: Questions 15-20 constitute the hand-graded portion of the exam. Questions 15 and 16 are worth 14 points each. Questions 17 - 20 are worth 15 points each. You **MUST** show all your work to receive full credit for these problems.

15. (14 pts) Evaluate the sum $\sum_{k=1}^{50} k(2k - 3)$. 
16. (14 pts) Consider a triangle ABC with sides $a = 2$ cm, $b = 4$ cm, and $c = 5$ cm.

(1) (7 pts) Calculate the angle C opposite side $c$. (If you do not have a calculator, leave an expression which can be directly evaluated on a calculator.)

(2) (7 pts) Find the area of the triangle ABC.
17. (15 pts) An observer on the ground measures the angle of elevation of the top of a tall building in Chicago to be 50°. He then walks 200 feet towards the building, and measures the angle of elevation of the top of the building to be 80°. What is the height of the building? (If you do not have a calculator, leave your answer as an expression which can be directly evaluated on a calculator.)
18. (15 pts) Find all solutions to the equation $\sin 2x + \sin 4x = 0$ over the interval $[0, 2\pi]$. 
19. (15 pts) Find all the complex solutions of the equation $3x^3 + 2x^2 + 2x - 1 = 0$. 
20. (15 pts) Find the general solution to the system of equations

\[
\begin{cases}
  x - y + 2z = 2 \\
  2x - y + z = 3 \\
  x - 2y + 5z = 3
\end{cases}
\]