READ AND FOLLOW THESE INSTRUCTIONS:
This booklet contains 17 pages, including this cover page. Check to see if any are missing. PRINT on the upper right-hand corner all the requested information, and sign your name. Put your name on the top of every page, in case the pages become separated. Do your work in the space provided in this booklet. There are two pages at the end of the exam that may be used as scratch. Show all your work. TEXTBOOKS, NOTES, AND CALCULATORS ARE NOT PERMISSIBLE.

INSTRUCTIONS FOR MACHINE-GRADED PART (Questions 1-8):
You MUST use a soft pencil (No. 1 or No. 2) to answer this part. Do not fold or tear the answer sheet, and carefully enter all the requested information according to the instructions you receive. DO NOT MAKE ANY STRAY MARKS ON THE ANSWER SHEET. When you have decided on a correct answer to a given question, circle the answer in this booklet and blacken completely the corresponding circle in the answer sheet. If you erase something, do so completely. Each question has a correct answer. If you give two different answers, the question will be marked wrong. There is no penalty for guessing, but if you don’t answer a question, skip the corresponding line in the answer sheet. Go on to the next question.

INSTRUCTIONS FOR THE HAND-GRADED PART (Questions 9-16):
Write your answers in the boxes which are provided. Show all work in the space provided below each problem. If you need extra space, state where the work is being done. Unsupported answers may receive little credit.

Notice regarding the machine graded sections of this exam: Either the student or the School of Mathematics may for any reason request a regrade of the machine graded part. All regrades will be based on responses in the test booklet, and not on the machine graded response sheet. Any problem for which the answer is not indicated in the test booklet, or which has no relevant accompanying calculations, will be marked wrong on the regrade. Therefore, work and answers must be clearly shown on the test booklet.

AFTER YOU FINISH BOTH PARTS OF THE EXAM: Place the answer sheet between two pages of this booklet (make a sandwich), with the side marked “GENERAL PURPOSE ANSWER SHEET” facing DOWN. Have your ID card in your hand when turning in your exam.

Multiple choice part (80 points) ______ Hand-graded part (120 points) ______

Total (200 points)______
PART I. Machine-graded problems - 80 points. (Each problem is worth 10 points.) Indicate which of the given choices is correct - in each case, only one answer is correct. Detailed justification is not required.

1. Which is correct for the solution of $P' = 3P - 6P^2$ with $P(0) = 1$?

(A) $\lim_{t \to \infty} P(t) = 0$

(B) $\lim_{t \to \infty} P(t) = 1/2$

(C) $\lim_{t \to \infty} P(t) = 2$

(D) $\lim_{t \to \infty} P(t) = \infty$

(E) $\lim_{t \to \infty} P(t)$ does not exist

2. In order to find a particular solution of $x'' - 7x' + 12x = \tan x$, one should

(A) Apply the method of undetermined coefficients.

(B) Apply variation of parameters.

(C) Use an integrating factor to combine the $x''$ and $x'$ terms.

(D) Linearize the equation around 0.

(E) Do a phase plane analysis.
3. Which is correct about the differential equation $x'' + 5x' + 2x = 0$?

(A) It is underdamped.

(B) It is critically damped.

(C) It is overdamped.

(D) None of these is correct.

4. Suppose you know that $e^{At} = \begin{bmatrix} 1+6t & 4t \\ -9t & 1-6t \end{bmatrix}$ for the system of linear equations $x' = Ax$.

What is $x(1)$ for the solution with $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$?

(A) $\begin{bmatrix} 11 \\ -14 \end{bmatrix}$

(B) $\begin{bmatrix} -11 \\ 14 \end{bmatrix}$

(C) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$

(E) $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$
5. Suppose $A, B,$ and $C$ are $3 \times 3$ matrices and consider the following statements:

(I) $\det(ABC) = \det(AB)\det(C)$,

(II) $\det(AB) = \det(BA)$,

(III) $\det(A) \neq 0$ implies $A$ is invertible,

(IV) $A$ invertible implies $\det(A) \neq 0$.

Then,

(A) None of these statements is correct.

(B) One of these statements is correct.

(C) Two of these statements are correct.

(D) Three of these statements are correct.

(E) All of these statements are correct.

6. Let $V$ be the set of all polynomials of degree 3 or less,
   let $W_1$ be the set of all polynomials of the form $at^3 + b(t - 1)^2$,
   let $W_2$ be the set of all polynomials of the form $t^2 + at + b$,
   let $W_3$ be the set of all polynomials of the form $a^2t^2$,
   where $a$ and $b$ can be any real numbers. Then,

(A) $W_1$ is a subspace of $V$, but $W_2, W_3$ are not.

(B) $W_2$ is a subspace of $V$, but $W_1, W_3$ are not.

(C) $W_3$ is a subspace of $V$, but $W_1, W_2$ are not.

(D) $W_1, W_2, W_3$ are all subspaces of $V$.

(E) None of the above is correct.
7. Which of the following sets of vectors is linearly independent? (Hint: look at all choices before you try to compute.)

(A) \{\langle1,1,2,3\rangle, \langle2,4,3,0\rangle, \langle1,0,1,2\rangle, \langle3,1,4,7\rangle\}

(B) \{\langle1,1,2,3\rangle, \langle2,4,3,0\rangle, \langle0,1,1,2\rangle, \langle1,4,5,9\rangle\}

(C) \{\langle1,1,2,3\rangle, \langle2,4,3,0\rangle, \langle0,0,1,2\rangle, \langle2,2,7,12\rangle\}

(D) \{\langle8,-6,7,5\rangle, \langle3,-6,7,0\rangle, \langle2,0,0,0\rangle, \langle9,11,0,0\rangle\}

(E) \{\langle2,4,1,2\rangle, \langle1,1,0,0\rangle, \langle1,0,0,0\rangle, \langle7,3,0,0\rangle\}

8. The critical point at \((0,0)\) of the system of equations

\[
\begin{align*}
x' &= 2x - y \\
y' &= -x + 2y
\end{align*}
\]

is a

(A) stable node,

(B) unstable node,

(C) saddle point,

(D) stable spiral,

(E) unstable spiral.
PART II. Hand-graded problems - 120 points. Write your answers in the boxes that are provided below each problem. If you need extra space, state where the work is being done. Also, be sure to justify your answers.

9. (15 pts) Solve the initial value problem

\[ y' + (\cos x)y = \cos x, \quad y(0) = 3. \]

Show all of your steps clearly.

Point total for page:
10. (10 pts) Let \( A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 1 & 2 \\ 6 & 0 & 3 \end{bmatrix} \) and \( B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \) be its inverse. Use adjoint matrices to find \( b_{12} \). Show your work.

How do you know \( B \) exists?

Point total for page:
11. (15 pts) What is the general solution of the differential equation

\[(D^4 + 5D^3 + 6D^2)x = 0\]

Show your work.
12. (15 pts) Find the general solution of the system of differential equations

\begin{align*}
x_1' &= x_1 + 2x_2, \\
x_2' &= 2x_1 + x_2.
\end{align*}

Show your work.
13. (10 pts) Let \( A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \). What is \( e^A \)? Show your work.
14. (25 pts) Let $A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. What is $A^{100}$? Clearly show all steps of your reasoning. (If the answer involves a high power of a constant that is not easy to compute, you can leave it in that form.) You can also use the next page for your work.
14. (continued)
15. (10 pts) What are the critical points of

\[x' = x - y - y^2,\]
\[y' = x^2 - 4y^2?\]

Show your work.
16. (20 pts) Consider the system of equations

\[
x' = x - 3y + 2xy,
\]
\[
y' = 4x - 6y - xy.
\]

You know that its critical points are at \((0, 0)\) and \((2/3, 2/5)\). What is the linearization at each of these two points? Carefully show your work.

at \((0, 0)\):

\[
\]

at \((2/3, 2/5)\):

\[
\]
SOME FORMULAS

\[ y_p = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} \, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} \, dx \]

\[ x(t) = \Phi(t) \Phi(a)^{-1} x_a + \Phi(t) \int_a^t \Phi(s)^{-1} f(s) \, ds \]

\[ x_1(t) = e^{pt}(a \cos qt - b \sin qt) \]

\[ x_2(t) = e^{pt}(b \cos qt + a \sin qt) \]

\[ A^{-1} = \frac{[A_{ij}]^T}{|A|} = \frac{[(-1)^{i+j} M_{ij}]^T}{|A|} \]