Math 2263
Spring 2003
FINAL EXAM

Name (Print)__________________________
Signature____________________________
Recitation Instructor______________________Section______I.D.#________________

READ AND FOLLOW THESE INSTRUCTIONS
This booklet contains 11 pages not including this cover page. Check to see if any are missing. PRINT on the upper right-hand corner all the requested information, and sign your name. Put your initials on the top of every page, in case the pages become separated. Standard calculators are allowed but textbooks and notes are not permissible. Do your work in the blank spaces and back of pages of this booklet. Show all your work.

There are 12 multiple choice problems each worth 15 points and 4 hand-graded problems each worth 30 points, for a total score of 300 points.

INSTRUCTIONS FOR MACHINE-GRADED PART (Questions 1-12):
You MUST use a soft pencil (No. 1 or No 2) to answer this part. Do not fold or tear the answer sheet, and carefully enter all the requested information according to the instructions you receive. Do not make any stray marks on the answer sheet. When you have decided on a correct answer to a given question, circle the answer in this booklet and blacken completely the corresponding circle in the answer sheet. If you erase something, do so completely. Each question has a correct answer. If you give two different answers, the question will be marked wrong. There is no penalty for guessing or skipping a particular problem.

INSTRUCTIONS FOR THE HAND-GRADED PART (Questions 13-16):
SHOW ALL WORK. Unsupported answers will receive little credit.

A note regarding the MACHINE graded sections of this exam: Either the student or the School of Mathematics may for any reason request a regrade of the machine graded part. All regrades will be based on responses in the test booklet, and not on the machine graded response sheet. Any problem for which the answer is not indicated in the test booklet, or which has no relevant accompanying calculations will be marked wrong on the regrade. Therefore, work and answers must be clearly shown on the test booklet.

AFTER YOU FINISH BOTH PARTS OF THE EXAM; Place the answer sheet between two pages of this booklet (make a sandwich), with the side marked “GENERAL PURPOSE ANSWER SHEET” facing DOWN. Have your ID card in your hand when turning in your exam.

Multiple choice part _______ Hand-graded part _______

Total ______

Letter Grade ______

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Multiple Choice Part

1. An equation for the tangent plane to the surface $x^2y - 2y^2z + 3xz^2 = 12$ at the point $(2, 1, -1)$ is

A. $z = 7(x - 2) + 8(y - 1) - 1$

B. $7x + 8y - 14z = 0$

C. $7x + 8y - 14z = 12$

D. $7x + 8y - 14z = 36$

E. $(2xy + 3z^2)(x - 2) + (x^2 - 4yz)(y - 1) + (-2y^2 + 6xz)(z + 1) = 0$

2. For curve $\vec{r}(t) = (e^t + e^{-t})\vec{i} + (e^t - e^{-t})\vec{j} + 2t \vec{k}$, a unit tangent vector at the point corresponding to $t = 1$ is

A. $\frac{1}{\sqrt{2}}(\vec{i} + \vec{k})$

B. $(e + \frac{1}{e})\vec{i} + (e - \frac{1}{e})\vec{j} + 2\vec{k}$

C. $\frac{1}{\sqrt{2(e^t + e^{-t})}}[(e + \frac{1}{e})\vec{i} + (e - \frac{1}{e})\vec{j} + 2\vec{k}]$

D. $\frac{1}{\sqrt{2(e^t + e^{-t})}}[(e - \frac{1}{e})\vec{i} + (e + \frac{1}{e})\vec{j} + 2\vec{k}]$

E. $\frac{\sqrt{2}}{e + \frac{1}{e}}[(e + \frac{1}{e})\vec{i} + (e - \frac{1}{e})\vec{j} + 2\vec{k}]$
3. Suppose \( u = x^3 + y^3 \) and \( x = e^{st} \) and \( y = e^{s-t} \). Then \( \frac{\partial u}{\partial t} \) at \( s = 2, \ t = 1 \) equals

A. \( e^6 + e^3 \)
B. \( 6e^6 - 3e^3 \)
C. \( 6e^6 + 3e^3 \)
D. \( 6e^4 - 3e^3 \)
E. \( 3e^6 - 3e^3 \)

4. Let \( f(x, y) = x^3 + \frac{1}{2} y^2 - 3xy + 9x - y \). Then

A. \( f \) has saddle points at both \((-1, -2)\) and \((-2, -5)\) and no other critical points
B. \( f \) has a local minimum at \((1, 4)\) and a local maximum at \((2, 7)\), and no other critical points
C. \( f \) has a saddle point at \((1, 4)\) and a local minimum at \((2, 7)\), and no other critical points
D. \( f \) has a saddle point at \((1, 4)\) and a local maximum at \((2, 7)\), and no other critical points
E. \( f \) has a local maximum at \((1, 4)\) and a local minimum at \((2, 7)\), and no other critical points
5. Changing the integration order in the iterated integral \( \int_0^2 \int_{y/2}^{5/2} f(x, y) \, dy \, dx \) leads to

- A. \( \int_0^1 \int_0^{2y} f(x, y) \, dx \, dy + \int_1^3 \int_0^{5-1} f(x, y) \, dx \, dy \)
- B. \( \int_0^1 \int_{2y}^{3-1} f(x, y) \, dx \, dy \)
- C. \( \int_0^2 \int_{2y}^{5-1} f(x, y) \, dx \, dy \)
- D. \( \int_0^1 \int_0^{2y} f(x, y) \, dx \, dy + \int_1^3 \int_{2y}^{5-1} f(x, y) \, dx \, dy \)
- E. \( \int_0^1 \int_0^{3/2} f(x, y) \, dx \, dy + \int_1^3 \int_0^{5/2} f(x, y) \, dx \, dy \)

6. Let \( C \) be the closed curve in \( \mathbb{R}^2 \) consisting of the segment on the \( y \)-axis joining \((0, -1)\) and \((0, 1)\), and of the semicircle \( x^2 + y^2 = 1 \) with \( x \geq 0 \) (also joining \((0, -1)\) and \((0, 1)\)), oriented counterclockwise. Then

\[
\oint_C (3x + 8y) \, dx + (3x - 7y) \, dy
\]

equals

- A. 0
- B. \(-\frac{5}{2} \pi\)
- C. \(\frac{5}{2} \pi\)
- D. \(-5 \pi\)
- E. \(5 \pi\)
7. Let $C$ be the curve in $\mathbf{R}^2$ which goes from the point $(3, 9)$ to $(2, 4)$ along the parabola $y = x^2$ and then continues on to $(0, 0)$ along the line $y = 2x$. The integral $\int_C 2xy \, dx + (x^2 + 2) \, dy$ equals

A. 180
B. $-180$
\[ \times \text{C. 99} \]
D. $-99$
E. 0

8. The solid $E$ consists of all points $(x, y, z)$ in $\mathbf{R}^3$ inside the cylinder $x^2 + y^2 = 4$ and outside the cone $x^2 + y^2 = z^2$, i.e., $E = \{(x, y, z) \mid x^2 + y^2 \leq 4 \text{ and } z^2 \leq x^2 + y^2\}$. Then in polar coordinates its volume is [note: the outline of the cone looks like the symbol “X” viewing horizontally]

A. $\int_0^{2\pi} \int_0^2 2r^2 \, dr \, d\theta$
B. $\int_0^{2\pi} \int_0^4 2r^2 \, dr \, d\theta$
C. $\int_0^{2\pi} \int_0^2 r^2 \, dr \, d\theta$
D. $\int_0^{2\pi} \int_0^2 2r \, dr \, d\theta$
E. $\int_0^{2\pi} \int_0^2 r \, dr \, d\theta$
9. A potential \( f \) for the vector field \( \vec{F} = (6x^2 - 5y^2 - 6xz^2, -10xy + 2z^3, -6x^2z + 6yz^2) \)
is (i.e. \( \vec{F} = \nabla f \))

- A. \((2x^3 - 5xy^2 - 3x^2z^2, -5xy^2 + 2yz^3, -3x^2z^2 + 2yz^3)\)
- B. \(2x^3 - 10xy^2 - 6x^2z^2 + 4yz^3\)
- C. \((12x - z^2, -10x, -6x^2 + 12yz)\)
- D. \(2x - 6x^2 + 12yz - 6x^2\)
- E. \(2x^3 - 5xy^2 - 3x^2z^2 + 2yz^3\)

10. Consider the region in the \( xy \)-plane bounded from above by the curve \( y = 4x - x^2 \) and from below by line \( y = x \). The centroid of this region (i.e., center-of-mass of this region for density = 1) is the point

- A. \((\frac{54}{5}, \frac{27}{4})\)
- B. \((\frac{12}{5}, \frac{3}{2})\)
- C. \((2, 4)\)
- D. \((\frac{9}{2}, \frac{12}{5})\)
- E. \((\frac{27}{4}, \frac{54}{5})\)
11. The volume of the region in the 1st octant bounded by the sphere \( x^2 + y^2 + z^2 = a^2 \), the cylinder \( x^2 + y^2 = a^2 \) and planes \( z = a, \ x = 0, \ y = 0 \) is given by

A. \[ \int_0^{\pi/2} \int_0^a \int_0^a 4 \sin \theta dzdrd\theta \]

B. \[ \int_0^{\pi/2} \int_0^a \int_0^a r^2 \sin \theta dzdrd\theta \]

C. \[ \int_0^{\pi/2} \int_0^a \int_0^a r dzdrd\theta \]

D. \[ \int_0^a \int_0^a \int_0^a dzdx dy \]

E. \[ \int_0^{\pi/2} \int_0^{\pi/4} \int_0^a \rho^2 \sin \varphi d\rho d\varphi d\theta \]

12. Which of the following statements is correct (assuming that all the functions involved have all orders of derivatives):

A. The divergence of a gradient field is always zero.

B. The divergence of a curl field can be nonzero if the domain is NOT simply connected.

C. The gradient of a divergence (of some vector field) is always a zero vector field.

D. The curl of a gradient field is always a zero vector field.

E. None of the above four statements is correct.
Hand-graded Part

13. (30pts) Find the maximum of \( f = 2x + 7y - 3z \) on the ellipsoid \( 2x^2 + 7y^2 + 3z^2 = 6. \)
14. (30pts) A parametric surface $S$ is given by $\vec{r} = \vec{r}(u, v)$:

$$x = u \cos v, \quad y = u \sin v, \quad z = v.$$ 

a) Find a unit normal vector $\vec{n}$ of the surface at a general point $\vec{r}(u, v)$. [Some intermediate computational steps here could be useful for b) as well.]

b) Suppose $S$ is defined by $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$. Calculate $I = \iint_S \sqrt{x^2 + y^2} \, dS$. 

15. (30pts) Consider a solid $E$ in $\mathbb{R}^3$ consisting of all points $(x,y,z)$ satisfying

$$z \geq x^2 + y^2 + z^2 - \sqrt{x^2 + y^2 + z^2}.$$

a) Apply the spherical coordinates $(\rho, \phi, \theta)$ to simplify this messy inequality description of the solid to some simpler expression like $\rho \leq \cdots$.

b) What are the ranges of the spherical coordinates for the solid?

c) Based on these results, calculate the volume $V$ of this solid.
16. (30 pts) Let $\mathbf{F} = (y - x)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$ and let $C$ be the boundary of the part of the plane $x + 2y + z = 2$ in the first octant oriented counterclockwise if viewed from point $(1,1,1)$, which is above the plane. Use Stokes' Theorem to evaluate $I = \oint_C \mathbf{F} \cdot d\mathbf{r}$. [Hint: notice that the plane could be treated as a graph surface.]